A Comparison of All Hexagonal and All Tetrahedral Finite Element Meshes for Elastic and Elasto-plastic Analysis

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ABSTRACT

This paper compares the accuracy of elastic and elastio-plastic solid continuum finite element analyses modeled with either all hexagonal or all tetrahedral meshes. Eigenvalues of element stiffness matrices, linear static displacements and stresses, dynamic modal frequencies, and plastic flow values in are computed and compared. Elements with both linear and quadratic displacement functions are evaluated. Linear incompressibility conditions are also investigated. A simple bar with a rectangular cross-section, fixed at one end, is modeled and results are compared to known analytical solutions wherever possible. The evaluation substantiates a strong preference for linear displacement hexagonal finite elements when compared solely to linear tetrahedral finite elements. The use of quadratic displacement formulated finite elements significantly improve the performance of the tetrahedral as well as the hexahedral elements. The nonlinear elasto-plastic comparison indicates that linear hexagonal elements may be superior to even quadratic tetrahedrons when shear stress in dominant. Results of this work may serve as a guide in selecting appropriate finite element types to be used in three dimensional elastic and elastic plastic analysis.

INTRODUCTION

Consideration of the convergence characteristics of two dimensional solutions of elastic continuum problems, using both quadrilateral and triangular elements, has been covered in previous studies and some finite element textbooks[1,2]. Such studies conclude that the significant factors that effect convergence characteristics of finite element solutions include the element's basic shape, element distortion, polynomial order of the element, completeness of polynomial functions, integration techniques, and material incompressibility. It is generally accepted that simplex triangular elements are inferior when compared to bilinear quadrilaterals. For example, statements such as "... for reasons of better accuracy and efficiency, quadrilateral elements are preferred for two-dimensional meshes and hexahedral elements for three-dimensional meshes. This preference is clear in structural analysis and seems to also hold for other engineering disciplines."[2] However, it also generally accepted that triangular elements, with higher order displacement assumptions, provide acceptable accuracy and convergence characteristics. However, mesh locking due to material incompressibility as reported by Hughes[3], is a serious shortcoming of triangular elements.

The current focus for developing rapidly converging finite element procedures is to incorporate h-p adaptive techniques.[4] Of particular note for this study is an article by Lo and Lee[5] which investigates the convergence of mixed element in h-p adaptive finite element analysis. A significant conclusion from this paper is, that by carefully controlling quality and grading, quadrilateral elements provide an increase in efficiency in h-p adaptivity over pure triangular elements.

A few studies have been published comparing the convergence characteristics of hexahedral verse tetrahedral meshes. Cifuentes and Kalbag [6] conclude that the results obtained with quadratic tetrahedral elements, compared to bilinear hexahedral elements, were equivalent in terms of both accuracy and CPU time. Bussler and Ramesh [7] report more accuracy using the same order hexahedral elements over tetrahedrons. Weingarten [8] indicates that both quadratic tetrahedrons and hexahedrons were equivalent in accuracy and efficiency and recommends using p method tetrahedrons to achieve desired accuracy. No studies were found incorporating incompressibility or plasticity aspects relating to the convergence of hexagonal and tetrahedral elements.

In this paper, stiffness matrix eigenvalues of a square geometrical volume, meshed with a single hexahedron is compared to the same geometrical volume meshed with five tetrahedrons. Next, results of a linear elastic, fixed end bar, meshed with either all hexahedrons or all tetrahedrons are compared. Both bending and torsional results are considered. The computed vibration modes of the fixed end bar problem are then evaluated. Finally, elasto-plastic calculations of the fixed end bar again meshed with both types of elements are evaluated.

STIFFNESS MATRIX EIGENVALUES

The evaluation of the eigenvalues and eigenvectors of a stiffness matrix is important when studying the convergence characteristics of any finite element.[9] Properly formulated elements have a zero valued eigenvalue associated with each rigid body motion. In addition, since the displacement based finite element technique overestimates the stiffness of a body, the smaller the eigenvalues for a particular deformation mode, the more effective is the element. Therefore, to provide an initial assessment of the effectiveness of simplex tetrahedrons compared with bilinear hexahedrons, the eigenvalues of equivalent models were computed. A regular unit cube volume, with a Young's Modulus of 30,000,000 and Poisson's Ratio of .3 was modeled with a single hexahedron and five tetrahedrons as shown in Figure 1. Note the configuration shown at the bottom of the figure 1 shows how the five tetrahedrons are positioned to fill the unit cube. The internal tetrahedron's position results in some directional properties of the stiffness matrix. The eigenvalues of the hexahedron were computed from (1) the stiffness matrix generated

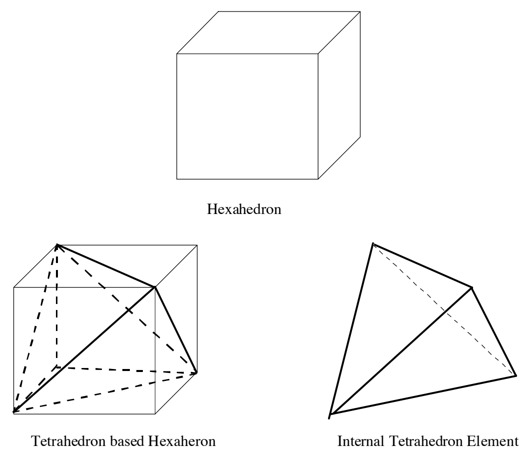


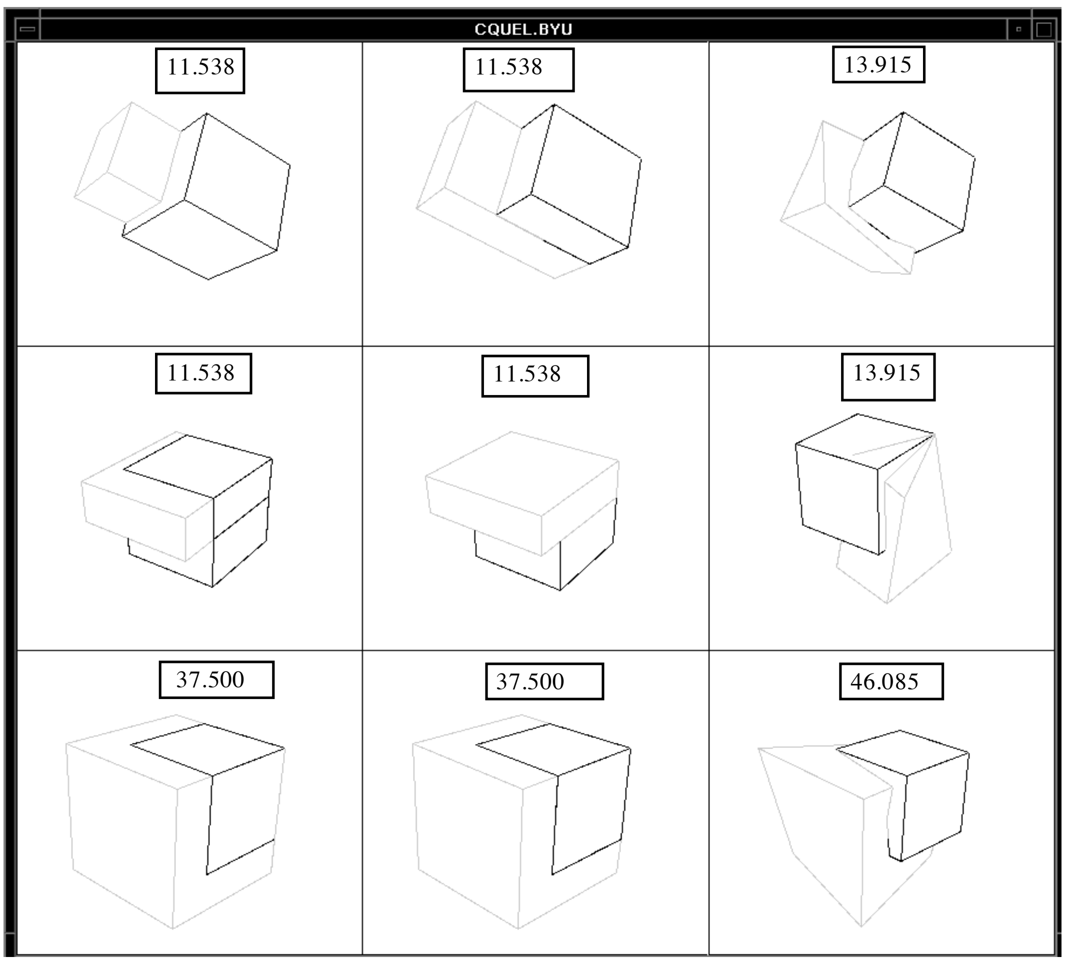
Figure 1. Regular Cube Region modeled with a Single Hexahedron (top) and Five Tetrahedrons (bottom)

from Nastran, (2) the stiffness matrix generated from a regular isoparametric 8 noded hexahedron using gaussian integration. The eigenvalues for the five tetrahedron condition were generated from combining the five tetrahedrons into the unit cube. The values of the computed eigenvalues are given in Table 1 where each value has a factor of 10^7.

Note that in all cases, the Nastran hexahedron always has the lesser and the 5 tetrahedron model always has the greater eigenvalue. The Nastran element is formulated using reduced or selective integration thus possessing some lower eigenvalues than the completely integrated isoparametric hexahedron. Three equivalent mode shapes (i.e. eigenvectors) with there associated computed strain energy (i.e. eigenvalues) are shown in Figure 2. Note the somewhat distorted mode shapes associated with the 5 tetrahedron element. This distortion is produced by the directional stiffness properties produced by the internal tetrahedron (see Figure 1).

Table 1. Stiffness Matrix Eigenvalues of unit cube modeled with (1) Nastran hexahedron, (2) Isoparametric hexahedron, and (3) 5 Simplex tetrahedrons

Eigenvalues	Nastran Hex	Isopar. Hex	5 Simplex Tets	Eigenvalues	Nastran Hex	Isopar. Hex	5 Simplex Tets
1	0	0	0	13	3.846	5.769	11.538
2	0	0	0	14	3.846	5.769	11.538
3	0	0	0	15	7.142	7.692	13.134
4	0	0	0	16	7.142	11.538	13.134
5	0	0	0	17	7.142	11.538	13.134
6	0	0	0	18	7.692	11.538	13.916
7	1.667	1.923	5.315	19	11.538	11.538	13.916
8	1.667	1.923	5.315	20	11.538	11.538	19.299
9	1.667	3.526	8.205	21	11.538	11.538	38.276
10	1.923	3.526	8.205	22	11.538	11.538	38.276
11	1.923	3.526	8.205	23	11.538	11.538	38.276
12	3.846	5.769	11.538	24	37.5	37.5	46.085



Nastran Hexahedron Simplex Hexahedron 5 Tetrahedron

Figure 2. Eigenvalues and Eigenvectors for three equivalent deformation modes

FINITE ELEMENT MODEL

A simple bar, fixed at one end, with a rectangular cross-section is used to compare the performance of linear and quadratic displacement assumption tetrahedral and hexahedral finite elements. The geometry, boundary conditions, and loading for this model are shown in Figure 3. The cases that were run are designated at LH - Linear Hexahedrons, QH - Quadratic Hexahedrons, LT - Liner Tetrahedrons, and QT - Quadratic Tetrahedrons. The finite element models are generated with either a regular 2x2 or 4x4 mesh across the cross-section of the bar as shown in Figure 3. Element length in the longitudinal direction is the same as shown in the cross-section view of Figure 3. The quadratic finite element model is generated by simply adding midside nodes to the linear model.

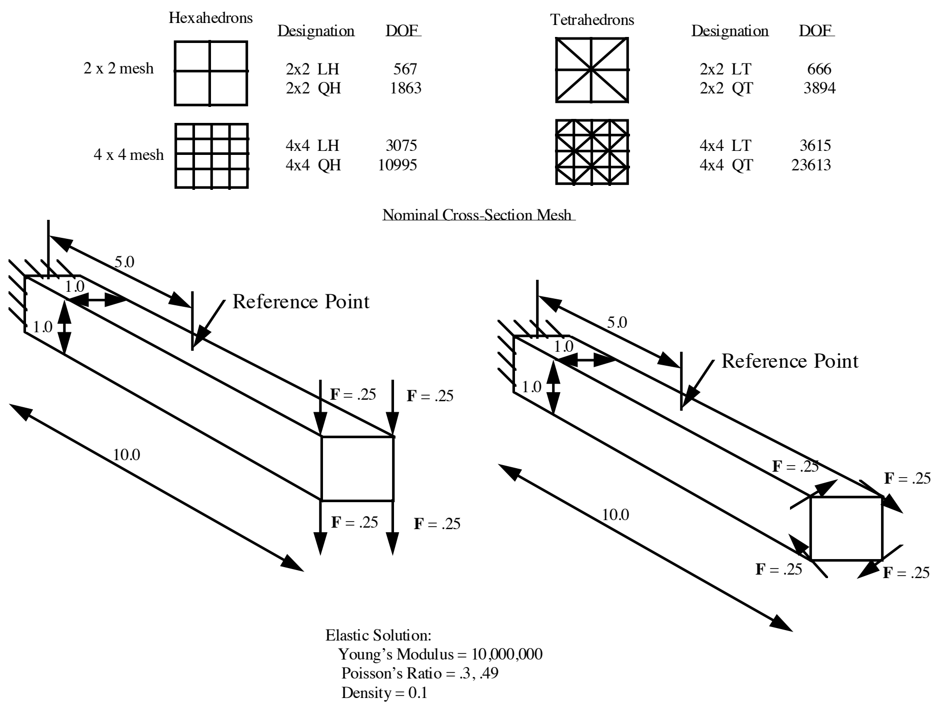


Figure 3. Basic model used for Static Bending and Torsional Analysis

STATIC LINEAR ANALYSIS

A cantilever beam with an end load and a torsion applied to the end of a fixed bar are the two structural cases used to compare the results of statically loaded all hexahedron and all tetrahedron meshes. For the bending problem, the analytical magnitudes of the normal displacement and the bending stress at the reference point, using classical beam theory [10] are 0.000125 and 30.0 respectively. Both the displacement and bending stress are independent of Poisson's ratio. For the torsional problem, the analytical magnitudes of the rotational displacement and the shear stress at the reference point were determined using the solution presented by Timoshenko and Goodier [11]. The shear stress from this solution is 6.8 and is independent of Poisson's ratio. The rotational displacement (i.e. the translation of the reference point in the direction of twisting) is 0.000003269 for a Poisson's ratio of .3, and 0.000003747 for a Poisson's ratio of .49. The errors, based on the above analytical solutions, are computed for the various finite element calculations and presented in Tables 2 and 3. Plots of the errors as a function of degrees of freedom in the finite element model are shown in Figures 4 and 5. These figures are plotted on log-log coordinates to more easily display the results over the large range of data.

TABLE 2. Error in Displacement and Stress at the Reference Position - Bending Model

Bending	ν = .3	Displacement			Bending	ν = .3	Stress		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	0.72%				567	0.00%			
666			31.48%		666			21.23%	
1863		0.24%			1863		0.01%		
3075	0.08%				3075	0.00%			
3615			10.48%		3615			21.00%	
3894				0.24%	3894				0.33%
10995		0.01%			10995		0.01%		
23613				0.01%	23613				0.01%
Bending	**ν = .49**	**Displacement**			**Bending**	**ν = .49**	**Stress**		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	6.56%				567	0.01%			
666			71.68%		666			66.77%	
1863		5.36%			1863		0.01%		
3075	3.20%				3075	0.01%			
3615			44.80%		3615			35.23%	
3894				4.80%	3894				0.10%
10995		2.88%			10995		0.01%		
23613				2.48%	23613				0.23%

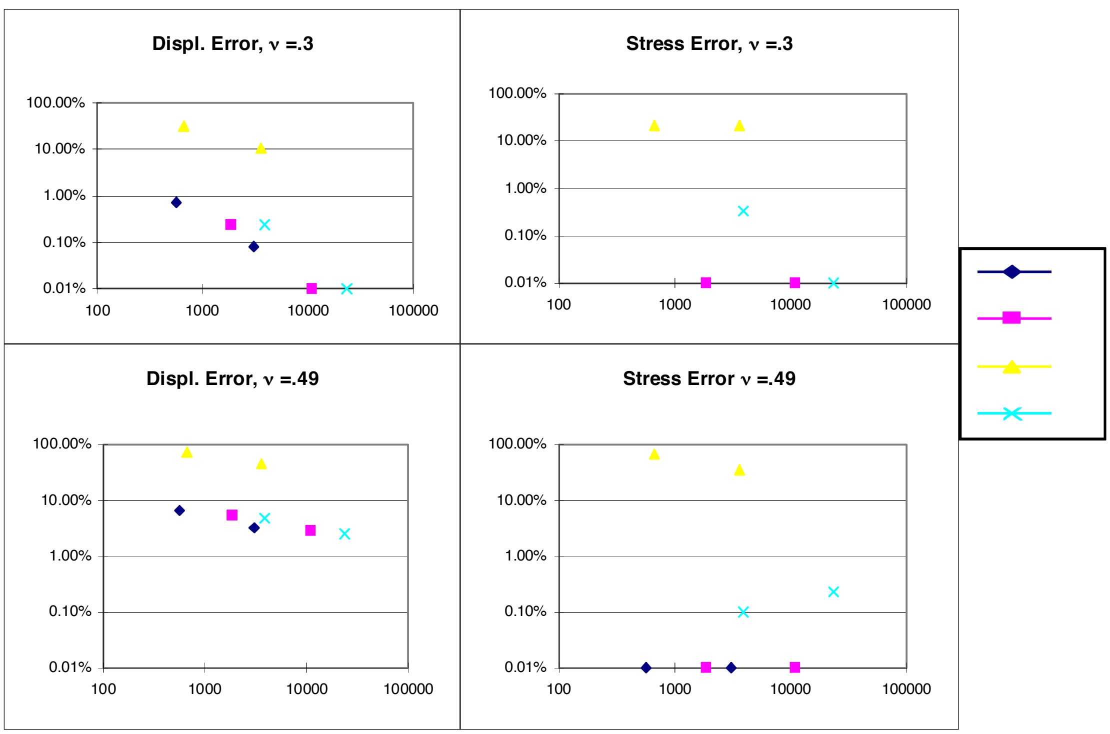


Figure 4. Log-Log plots, displacement and stress error vs. DOF - Bending Calculations

TABLE 3. Error in Displacement and Stress at the Reference Position - Torsion Model

Bending	ν = .3	Displacement			Bending	ν = .3	Stress		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	15.65%				567	37.59%			
666			50.81%		666			77.82%	
1863		1.99%			1863		7.97%		
3075	5.26%				3075	8.59%			
3615			22.39%		3615			38.40%	
3894				3.32%	3894				0.07%
10995		0.49%			10995		0.01%		
23613				0.76%	23613				0.01%
Bending	**ν = .49**	**Displacement**			**Bending**	**ν = .49**	**Stress**		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	26.41%				567	26.41%			
666			68.80%		666			68.80%	
1863		2.60%			1863		2.60%		
3075	5.44%				3075	5.44%			
3615			52.72%		3615			52.72%	
3894				4.70%	3894				4.70%
10995		0.75%			10995		0.75%		
23613				1.41%	23613				1.41%

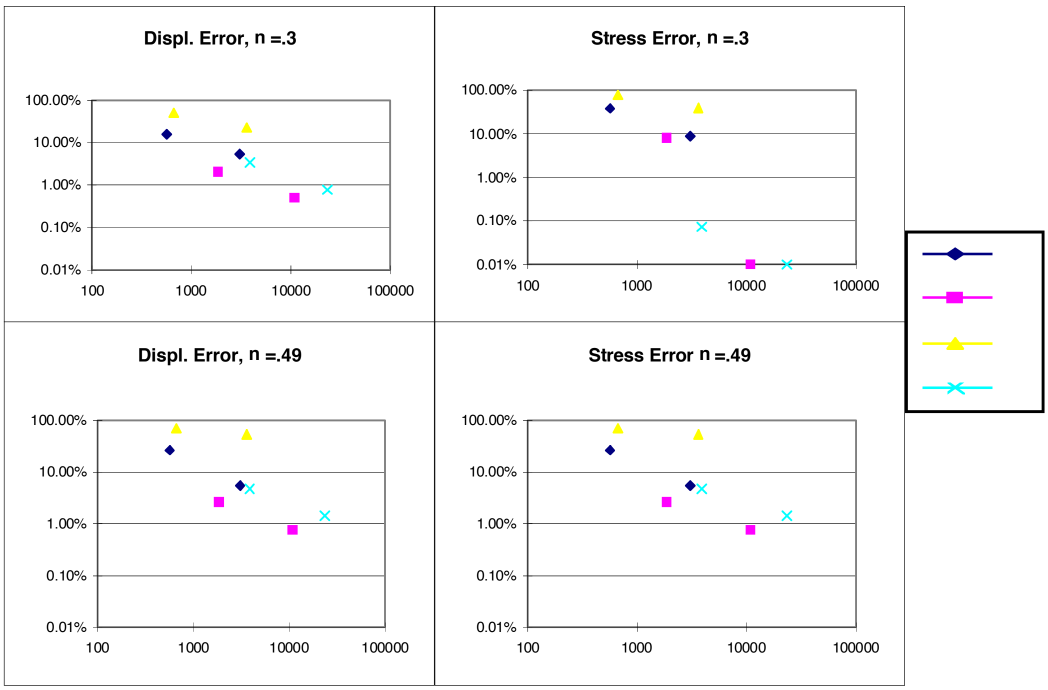


Figure 5. Log-Log plots, displacement and stress error vs. DOF - Torsion Calculations

Note that in all cases, the linear tetrahedron element (LT) produces the maximum error. The performance of the linear hexahedron (LH) is significantly enhanced in the bending problem because the LH element is formulated using a selective integration technique.[12] However, the performance of the LH element is still superior to the LT in the torsion solution when selective integration is not an issue. The figures clearly demonstrate that the quadratic elements perform adequately in all cases with the quadratic hexahedron (QH) showing slightly better performance over the quadratic tetrahedron (QT). Note the degradation in all solutions as Poisson's ratio approaches 0.5. P refinement characteristics can also be deduced from Tables 2 and 3 and Figures 4 and 5. The quadratic models for both the hexahedron and tetrahedron meshes are generated by simply adding mid-side nodes to the original linear element model. For example, by associating the first LH data point with the first QH data point, LH p refinement convergence is displayed.

DYNAMIC MODAL ANALYSIS

The natural modes of vibration are compared in this section. Again the bar shown in Figure 3 is used with the same meshing models used for the static analysis. The analytical solution for the bending mode is given by Hurty and Rubenstien [13] as 317.5 cycles/sec. An approximate solution for the torsional vibration mode, assuming the stiffness value as determined from the elasticity solution [11], and no warping, is 2614 cycles/sec.

TABLE 4. Error in Eigenvalues for Bending and Torsional Vibration

Bending	ν = .3	Frequency			Torsion	ν = .3	Frequency		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	0.06%				567	8.86%			
666			20.28%		666			41.68%	
1863		0.09%			1863		0.98%		
3075	0.19%				3075	2.82%			
3615			0.28%		3615			0.36%	
3894				0.13%	3894				1.64%
10995		0.28%			10995		0.21%		
23613				0.28%	23613				0.36%
Bending	ν = .49	Frequency			Torsion	ν = .49	Frequency		
DOF	LH	QH	LT	QT	DOF	LH	QH	LT	QT
567	2.68%				567	8.88%			
666			75.12%		666			26.32%	
1863		2.08%			1863		1.31%		
3075	1.20%				3075	2.82%			
3615			23.87%		3615			29.76%	
3894				1.83%	3894				2.41%
10995		1.04%			10995		0.35%		
23613				0.85%	23613				0.69%

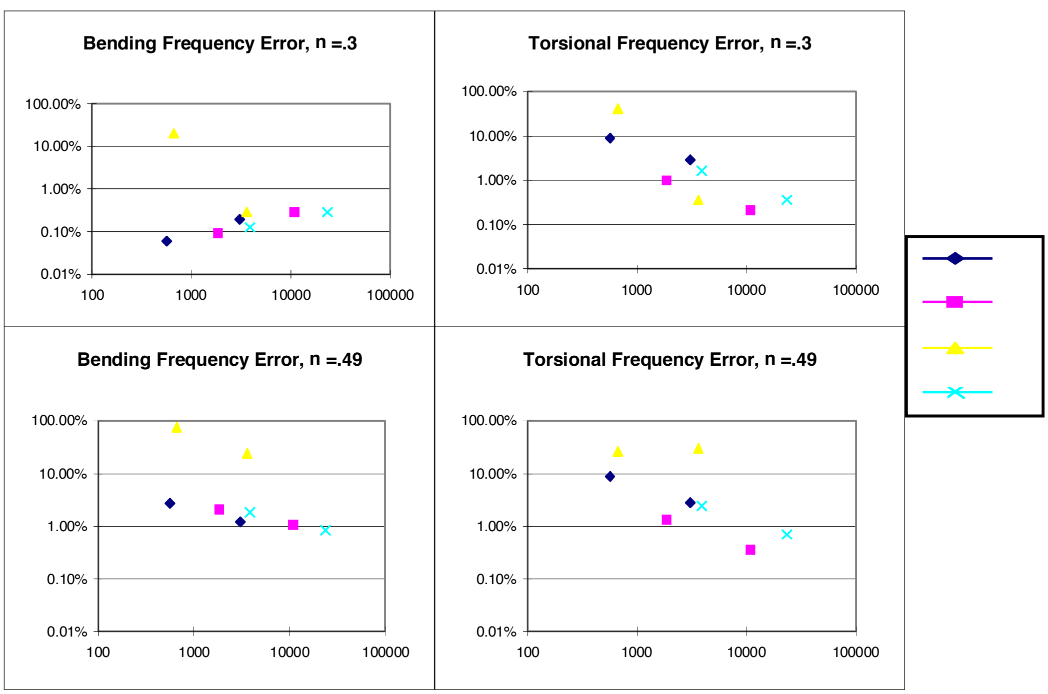


Figure 6. Log-Log plots, displacement and stress error vs. DOF - Bending Calculations

Comparison of the results in Table 4 and Figure 6 lead to the same conclusions as before. The linear tetrahedron performs poorly in all cases. The quadratic elements are adequate with a slight advantage given to the hexagonal mesh.

STATIC NONLINEAR ELASTO-PLASTIC ANALYSIS

In this section a comparison of all hexahedral and all tetrahedral meshes is made for static nonlinear, elasto-plastic calculations. Only material nonlinearity is investigated. Again a bending and torsion cases, as shown in Figure 3, are evaluated. The nonlinear behavior of the material is assumed to be elastic perfectly plastic with Young's Modulus of 10,000,000., Poisson's Ratio of .3, and Yield Stress of 10,000. A nonlinear analysis is much more difficult to compute than a linear analysis and comparison of results is dependent not only on element discritization but nonlinear solution techniques. A complete discussion on the process of solving nonlinear elastio-plastic problems is given in Reference 9.

The analytical bending solution used for the comparison basis is depicted in Figure 7. Here the beam is loaded elasticity until incipient plasticity, at the upper and lower surfaces of the supported end, is attained. The load is then increased incrementally and the plastic region continues to grow and propagate. The nonlinear convergence criteria was set to .1% on the strain energy norm. The displacement of the loaded tip of the beam and the distance, D, (i.e. the distance to the incipient yield front on surface of the beam), are quantities used for comparison between all hexahedral and all tetrahedral meshes.

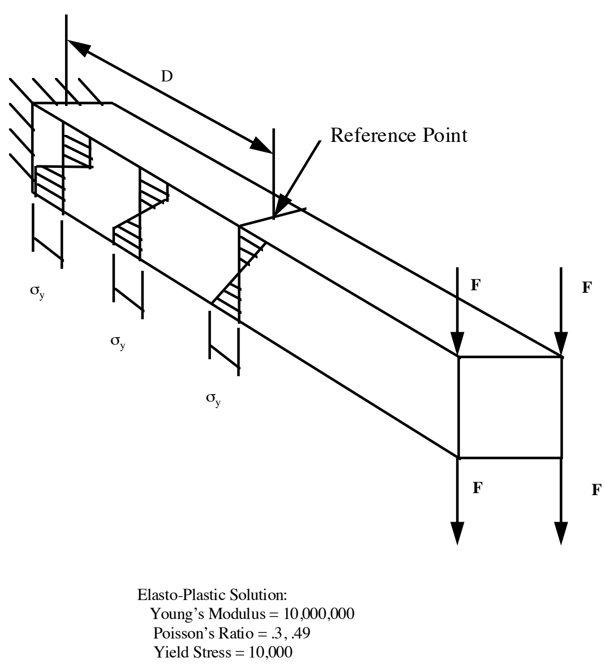


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Figure 7. Elasto-Plastic Bending Solution

Table 5 and Figure 8 show the results of the calculations of the nonlinear bending comparison problem. Note that the results compare very favorably for all calculations except for the 2x2 and 4x4 linear tetrahedron models. The D values had similar accuracy as the displacement values shown in Table % and Figure 8. The D values, in all cases except for the 2x2 and 4x4 linear tetrahedron models, compared almost exactly with the analytical solution. The linear tetrahedron models did not even manifest any plasticity except for the last two loading steps on the 4x4 mesh. Thus the D values were either 0 or very small.

TABLE 5. Tip Displacement from nonlinear calculations

	Tip Displacement					
Step	0	1	2	3	4	5
Load, 4xF	0	160	185.2	196.08	208.3	238
2x2 LH	0	0.0638	0.0739	0.0782	0.0831	0.1020
2x2 QH	0	0.0641	0.0742	0.0786	0.0845	0.1040
4x4 LH	0	0.0642	0.0743	0.0787	0.0847	0.1040
4x4 QH	0					
2x2 LT	0	0.0407	0.0471	0.0499	0.0530	0.0606
2x2 QT	0	0.0641	0.0741	0.0785	0.0837	0.1020
4x4 LT	0	0.0569	0.0658	0.0697	0.0744	0.0875
4x4 QT	0	0.0640	0.0740	0.0786	0.0848	0.1020

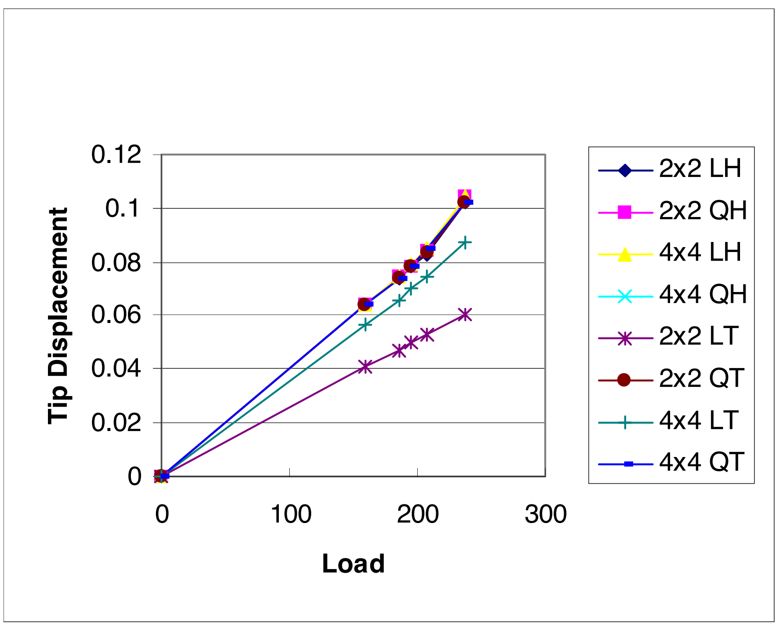

Figure 8. Tip Displacements as a function of applied loads - nonlinear bending example

The analytical work used for the comparison basis for the torsion problem is a finite difference solution given by Mendleson. [14] Here, as in the elastic comparison, the translation of the reference point in the direction of twisting is the result used for comparison. Table 6 and Figure 9 show the results of the calculations of the nonlinear torsion comparison problem.

TABLE 6. Mid-point displacements from nonlinear calculations

	Mid-point Displacement				
Step		1	2	3	4
Applied Torque	0	1201	1717	1844	1880
2x2 LH	0	0.00468	0.0067	0.00795	0.0083
2x2 QH	0	0.00547	0.00914	0.0108	0.0117
4x4 LH	0	0.00527	0.0083	0.01	0.0107
4x4 QH	0				
2x2 LT	0	0.00272	0.0039	0.00419	0.00427
2x2 QT	0	0.00537	0.00769	0.00825	0.00841
4x4 LT	0	0.00425	0.00609	0.00654	0.00667
4x4 QT	0	0.00553	0.00791	0.0085	0.00866
Ref 14	0	0.00555	0.01055	0.01305	0.01555

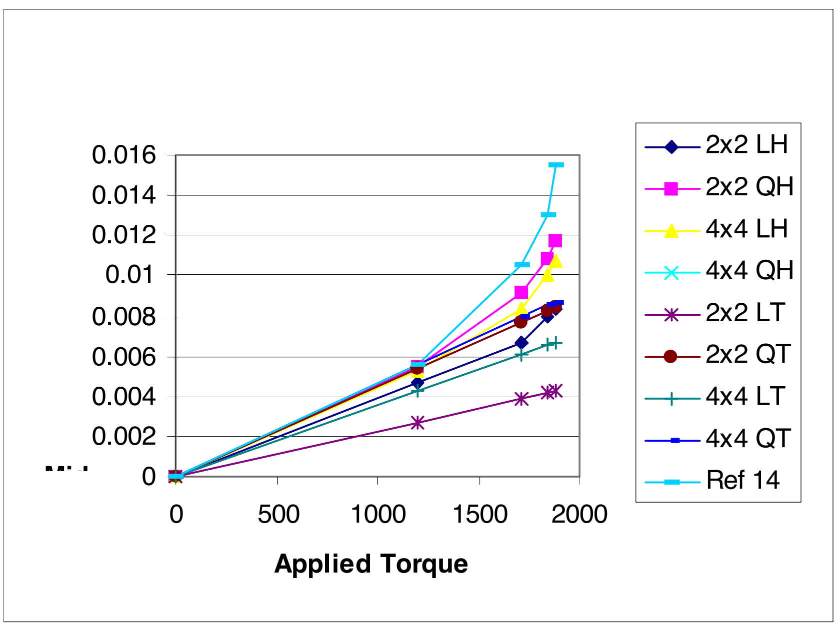

Figure 9. Mid point displacements as a function of applied torque - nonlinear torsion

Similar to the elastic calculations, the nonlinear torsion problem manifests significantly more error than does the bending problem. This problem has a significant region of plastic flow within the continuum volume, thus requires much more nonlinear iterative calculations than were required in the bending solution. In fact, the calculations would not converge using the .1% strain energy norm that was used for the bending problem. To allow for convergence, the nonlinear convergence criteria had to be weakened to a 5% displacement norm in all cases except for the 2x2 LH model. This criteria may not be a tight enough to provide an acceptably accurate solution. Note the sharper bend in the 2x2 LH model on the final step that results from using the tighter convergence criteria.

Note the significantly greater accuracy that comes from the hexagonal models when compared to the tetrahedral models. Again, similar to the bending calculations, the LH models did not even initiate any plasticity even at the maximum loading.

CONCLUSIONS

Numerous calculations have been conducted in this paper that compare the accuracy of all tetrahedral meshes to all hexahedral meshes. First it was shown that the stiffness matrix eigenvalues for linear tetrahedrons were generally larger that those for linear hexahedrons. This fact demonstrates that linear hexahedrons can generally deform in a lower strain energy state, thus making them more accurate than linear tetrahedrons in numerous situations. The eigenvalue analysis was not conducted for quadratic elements.

The comparison of linear static bending situation indicated that LT models produced errors between 10 to 70 percent in both displacement and stress calculations. Such errors are obviously unacceptable for stress analysis work. However LH, QH, and QT models all provided acceptable results, even with relatively coarse meshes. In all cases, the error was significantly greater with a nearly incompressible material model (i.e. $\nu = .49$).

The linear static torsion problem again showed that the LT element produced errors of an unacceptable magnitude. This problem also demonstrated that, because selective integration is only effective on the bending problem, the LH element, without a significant number of degrees of freedom, produces poor results. Here, as in the previous problem, the QH element is superior.

Dynamic modal analysis comparisons reflected the results of the previous two comparisons. That is, LT models produce more error that in acceptable and QH models are generally preferred to insure accuracy with both torsion and nearly incompressible materials.

Significant information is conspicuous in the nonlinear elasto-plastic calculations. Here, as before, all but LT models are adequate for bending calculations. However, not only LT, but QT models seemed to underperform both LH and QH elements. More study is required to determine why QT elements appear to underperform LH elements in significant plastic flow calculations.

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